

Weak Lie symmetry and extended Lie algebra

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Abstract:¹

The concept of weak Lie motion (weak Lie symmetry) is introduced through $\mathcal{L}_\xi \mathcal{L}_\xi g_{ab} = 0$, $(\mathcal{L}_\xi \mathcal{L}_\xi f = 0)$. Applications are given which exhibit a reduction of the usual symmetry, e.g., in the case of the the rotation group. In this context, a particular generalization of Lie algebras is found (“extended Lie algebras”) which turns out to be an involutive distribution or a simple example for a tangent Lie algebroid. Riemannian and Lorentz metrics can be introduced on such an algebroid through an extended Cartan-Killing form. Transformation groups from non-relativistic mechanics and quantum mechanics lead to such tangent Lie algebroids and to Lorentz geometries constructed on them (1-dimensional gravitational fields).

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1 Introduction

In 1872 Felix Klein formulated his Erlangen program as such: “A manifold is given and with it a group of transformations. [...] Develop the theory of invariants with regard to this group” ([1], p. 28). According to him, Sophus Lie accepted this program and spread it among his students.²

In the first part of what follows, in the spirit of Klein, several new concepts will be introduced and investigated: *weak (Lie) motions* (cf. section 4) and *groups of extended motions* (cf. section 8). The latter concept is related to a suggested widening of the physicists’ concept of a Lie algebra to particular tangent Lie algebroids (*extended Lie algebras*). Some of the corresponding finite transformations forming groups are presented: they are no longer Lie groups. I will also propose an extension of the Cartan-Killing form which up to now seemingly has not been studied. Its definition allows the introduction of Riemannian and Lorentz metrics on the sections of a subbundle of the tangent bundle. The mathematical literature for algebroids and groupoids (eg., [5]), has lead to a few formal applications to Lagrangian mechanics [6], [7], [8]. The particular tangent Lie algebroids presented here are an example for such structures much closer to physics than the examples usually given by mathematicians.

2 Lie-dragging

2.1 Preliminaries

In metric geometry, the concept of symmetry may be expressed by an isometry of the metrical tensor g_{ab} of such a space. This means that this tensor field remains unchanged along the flow of a vector field X . An expression for this demand may be formulated by help of the Lie derivative defined for tangent vector fields $X := \xi^a \frac{\partial}{\partial x^a}$, $Y := \eta^a \frac{\partial}{\partial x^a}$ by:

$$\mathcal{L}_X Y = [X, Y], \quad (1)$$

where $[\cdot, \cdot]$ denotes the Lie-bracket $[A, B] = AB - BA$. If (1) is expressed by the components ξ^a, η^a of the tangent vectors X, Y , then

$$\mathcal{L}_\xi \eta^a = \eta^a_{,c} \xi^c - \eta^c \xi^a_{,c}, \quad (2)$$

²The history of the “Erlanger Programm” is much more complicated, though, cf. [2], [3]. For the importance of Kleins ideas for physics cf. [4].

where $\eta^a_{,c} = \frac{\partial \eta^a}{\partial x^c}$. If $\mathcal{L}_X Y = 0$, the vector field X is called a symmetry of the vector field Y .³ The Leibniz rule holds for the Lie derivative.⁴ From (1) we have

$$\mathcal{L}_Z \mathcal{L}_X Y = [Z, [X, Y]], \quad (3)$$

and with help of the Jacobi identity:

$$\mathcal{L}_Z \mathcal{L}_X Y + \mathcal{L}_Y \mathcal{L}_Z X + \mathcal{L}_X \mathcal{L}_Y Z = [Z, [X, Y]] + [Y, [Z, X]] + [X, [Y, Z]] = 0. \quad (4)$$

From (4):

$$\mathcal{L}_Z \mathcal{L}_X Y - \mathcal{L}_X \mathcal{L}_Z Y = [[X, Z], Y] = \mathcal{L}_{[X, Z]} Y = \mathcal{L}_{\mathcal{L}_X Z} Y. \quad (5)$$

For a Lie group, a special subspace of the tangent space is formed by the infinitesimal generators $X_{(i)} := \xi_{(i)}^a \frac{\partial}{\partial x^a}$, ($i, j, l = 1, 2, \dots, p$) of a Lie-algebra

$$[X_{(i)}, X_{(j)}] = c_{ij}^l X_{(l)}, \quad (6)$$

with structure constants⁵ c_{ij}^l . From (6) we obtain:

$$\mathcal{L}_{X_i} \mathcal{L}_{X_j} X_k = c_{jk}^l c_{il}^m X_m \quad (7)$$

such that according to (4):

$$c_{jk}^l c_{il}^m + c_{ij}^l c_{kl}^m + c_{ki}^l c_{jl}^m = 0. \quad (8)$$

A symmetric bilinear form, the Cartan-Killing form, may be introduced:

$$\sigma_{ij} := c_{il}^m c_{jm}^l. \quad (9)$$

³Such symmetries play an important role for the integration of differential equation, cf. [10].

⁴Latin indices from the beginning (a, b, c, \dots) and end of the alphabet (r, s, t, \dots) run from 1 to n or 0 to $n-1$ where n is the dimension of the space considered. Indices from the middle (i, j, k, l, \dots) may take other values. The summation convention is used except when indicated otherwise.

⁵In current mathematical literature, the definition of a Lie algebra is much more general. It is defined either as a module $\mathcal{B}(M)$ of the set of all C^∞ -vector fields on a C^∞ -manifold with a multiplication introduced via the Lie-bracket, or as a finite-dimensional vector space V over the real or complex numbers with a bilinear multiplication on it defined by an anti-commuting bracket $[\cdot, \cdot]$ satisfying the Jacobi identity (4).

If it is nondegenerate, i.e., for semisimple Lie groups, σ_{ij} can be used as a metric in group space.

In section 7, we will permit that the structure constants become directly dependent on the components ξ_i^a of the vector fields $X_i(x)$: they will become *structure functions*.⁶

2.2 Lie-dragging (with examples)

Under “Lie-dragging” with regard to an arbitrary C^∞ vector field $X = \xi^a \frac{\partial}{\partial x^a}$ we understand the operation of the Lie derivative on any geometric object *without* the simultaneous requirement that the result be zero.⁷ Applied to the metric g_{ab} , this means

$$\mathcal{L}_\xi g_{ab} = \gamma_{ab} , \quad (10)$$

where γ_{ab} is a symmetric tensor of any rank between 0 and n (in n -dimensional space). In the sequel we will be interested in the case $\gamma_{ab} \neq \lambda g_{ab}$.

For a tensor field, Lie-dragging neither conserves the rank of the field, nor, if it is exerted on a symmetric bilinear form, its signature. The quest for the conditions that Lie-dragging leads to a specific rank or specific signature of a tensor field could be among the first mathematical investigations into the concept (with rank 0 of γ_{ab} being set aside). Also, the vector fields X might be classified according to whether Lie-dragging with them leads to a prescribed rank for given metric g_{ab} . In any case, not every arbitrary γ_{ab} can be reached by Lie-dragging (cf. Appendix 1).

Equation (10) can be read in different ways:

A) Given a single vector field (a set of vector fields) *and* an arbitrary metric g_{ab} ; the set of all possible bilinear forms γ_{ab} is to be determined by a straightforward calculation. This is an intermediate step for the determination of

⁶The *structure constants* in (6) are brought into the definitions of a Lie algebra presented in the previous footnote by the choice of a basis $\{Y_1, Y_2, \dots, Y_n\}$ of V . The multiplicative action is determined for all vectors X, Y of V only if all brackets $[X, Y]$ are known. According to one author: “We ‘know’ them by writing them as linear combinations of the Y_i . The coefficients c_{ij}^l in the relations $[Y_i, Y_j] = c_{ij}^l Y_l$ are called structure constants” ([9], pp. 1, 5). This recipe no longer works for vector fields which cannot be generated by linear combinations with constant coefficients from a basis. Cf. section 7.

⁷This use of the name “Lie-dragging” is different from the one in [12]. By (1), the Lie-dragging of a vector field is expressed.

weak Lie motions of g_{ab} .

B) Given a single vector field (a set of vector fields) and a fixed target tensor γ_{ab} ; the metrics g_{ab} which are Lie-dragged into it are to be determined. This requires solving a system of 1st-order PDEs.

C) Given both a start metric g_{ab} and a target metric γ_{ab} . The task is to determine the vector fields X dragging the one into the other.⁸

For a first example for Lie-dragging in space-time leading to tensors of lower rank, we look at the Kasner metric:

$$ds^2 = (dx^0)^2 - (x^0)^{2p_1}(dx^1)^2 - (x^0)^{2p_2}(dx^2)^2 - (x^0)^{2p_3}(dx^3)^2, \quad (11)$$

an exact solution of Einstein's vacuum field equations if $p_1 + p_2 + p_3 = 1 = (p_1)^2 + (p_2)^2 + (p_3)^2$, p_1, p_2, p_3 constants. Lie-dragging with

$$X = \delta_0^a \frac{\partial}{\partial x^a}$$

leads to a bilinear form of rank 3, i.e., after a coordinate change, to the space sections:

$$ds^2 = -(y^0)^{2p_1}(dy^1)^2 - (y^0)^{2p_2}(dy^2)^2 - (y^0)^{2p_3}(dy^3)^2.$$

Unlike this, Lie-dragging of (11) with

$$X = f(x^0) \delta_1^a \frac{\partial}{\partial x^a}$$

leads to a tensor of rank 2: $\gamma_{ab} = 2 \frac{df(x^0)}{dx^0} g_{1(a} \delta_{b)}^0$.

In the second example, a Lie-dragged metric of rank 1 is prescribed. Let

$$\mathcal{L}_\xi g_{ab} = X_a X_b, \quad (12)$$

with the vector field X tangent to a null geodesic:

$$(\overset{g}{\nabla}_b X_a) X^b = 0, \quad g_{ab} X^a X^b = 0. \quad (13)$$

From the definition of $\mathcal{L}_\xi g_{ab}$ given in (16) and (13), $(X^s \xi_s)_{,a} X^a = 0$ follows: $X^s \xi_s$ must be constant along the geodesic. (13) leads to a restriction on ξ for given null geodesic, or for X^a if the vector field ξ is given. X^a generates a super-weak motion (cf. section 4).

The collineations presented in section 3 are also examples for Lie-dragging.

⁸If we ask for both, $\mathcal{L}_X g_{ab} = \gamma_{ab}$ and $\mathcal{L}_X \gamma_{ab} = g_{ab}$, then we are back to weak homothetic mappings for both g and γ . Cf. next section.

3 Motions and Collineations

On a manifold with differentiable metric structure, a motion is defined by the vanishing of the Lie-derivative of the metric with regard to the tangent vector field $X = \xi^a \frac{\partial}{\partial x^a}$:

$$\begin{aligned}\mathcal{L}_X g(Y, Z) &= 0 = Xg(Y, Z) + g(Z, \mathcal{L}_X Y) + g(Y, \mathcal{L}_X Z) \\ &= Xg(Y, Z) + g(Z, [X, Y]) + g(Y, [X, Z]),\end{aligned}\quad (14)$$

where X, Y, Z are tangent vector fields. In local coordinates, (14) reads as:

$$\gamma_{ab} = \mathcal{L}_\xi g_{ab} = 0 = g_{ab,c} \xi^c + g_{cb} \xi^c_{,a} + g_{ac} \xi^c_{,b}, \quad (15)$$

with $g_{ab} = g_{ba}$. The vector field ξ is named a *Killing vector*; its components generate an infinitesimal symmetry transformation:⁹ $x^i \rightarrow x^{i'} = x^i + \xi^i$. (15) may be expressed in a different form:¹⁰

$$\mathcal{L}_\xi g_{ab} = 2\overset{g}{\nabla}_{(a}\xi_{b)} = 0. \quad (16)$$

In (16), $\overset{g}{\nabla}$ is the covariant derivative with respect to the metric g_{ab} (Levi Civita connection), and $\xi_a = g_{ab}\xi^b$. From (15) we can conclude that $\mathcal{L}_\xi ds = 0$ for all dx^a , i.e., all distances remain invariant. A consequence of (15) is that the motions ξ form a Lie group and the corresponding infinitesimal generators $X_{(i)} := \xi^{\sigma}_{(i)} \frac{\partial}{\partial x^{\sigma}}$ a Lie algebra (6) (cf. [22]).

As an example for a group of motions in 3-dimensional Euclidean space, we start from a Lie group G_3 acting on V_3 with finite transformations:

$$x^{1'} = x^1 + c_1, \quad x^{2'} = x^2 + c_2 x^1, \quad x^{3'} = x^3 + c_3. \quad (17)$$

The corresponding Lie algebra is ([13], p. 213):

$$[X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = X_1. \quad (18)$$

⁹For mechanical systems in phase space, this infinitesimal symmetry transformation is applied to the generalized coordinates and supplemented by an infinitesimal transformation for the momenta: $p_a \rightarrow p_{a'} = p_a + \eta_a$ with an additional infinitesimal generator η_a . Cf. [11]. The authors use the name “weak-Lie” symmetry for what we would name Lie symmetry.

¹⁰Symmetrization brackets are used: $A_{(r}B_s) = \frac{1}{2}(A_r B_s + A_s B_r)$; $A_{[r}B_s] = \frac{1}{2}(A_r B_s - A_s B_r)$.

Lie-dragging with the vector fields $\xi_1^a = \delta_2^a$, $\xi_2^a = \delta_3^a$, $\xi_3^a = -\delta_1^a + x^3 \delta_2^a$ gives:

$$\begin{aligned}\mathcal{L}_{\xi_1} g_{ab} &= g_{ab,2} =: \gamma_{ab}^{(1)}, \quad \mathcal{L}_{\xi_2} g_{ab} = g_{ab,3} =: \gamma_{ab}^{(2)}, \\ \mathcal{L}_{\xi_3} g_{ab} &= -g_{ab,1} + x^3 g_{ab,2} + 2g_{2(a} \delta_{b)}^3 =: \gamma_{ab}^{(3)}.\end{aligned}\tag{19}$$

All $\gamma_{ab}^{(i)}$ can have full rank. The demand $\gamma_{ab}^{(i)} = 0$, $i = 1, 2, 3$, makes this G_3 a group of motions whence follows:

$$g_{ab} = \begin{pmatrix} \alpha_{11}^{(0)} & \alpha_{12}^{(0)} & P_1 \\ \alpha_{21}^{(0)} & \alpha_{22}^{(0)} & P'_1 \\ P_1 & P'_1 & P_2 \end{pmatrix},$$

where $P_1 = \alpha_{12}^{(0)} x^1 + \alpha_{13}^{(0)}$, $P'_1 = \alpha_{22}^{(0)} x^1 + \alpha_{23}^{(0)}$ and $P_2 = \alpha_{22}^{(0)} (x^1)^2 + 2\alpha_{23}^{(0)} x^1 + \alpha_{33}^{(0)}$ with $\alpha_{33}^{(0)}, \alpha_{1p}^{(0)}, \alpha_{2p}^{(0)}$, ($p = 1, 2, 3$) constants. We will see in section 5.3 how the metric looks if the group is demanded to be a complete set of *weak* (Lie) motions.

Further types of symmetries are defined by the vanishing of the Lie derivative applied to other geometric objects like connection (“affine collineations” $\mathcal{L}_\xi \Gamma_{ab}^c(g) = 0$, cf. [14]), curvature tensor (“curvature collineations” $\mathcal{L}_\xi R_{dab}^c(g) = 0$, cf. [15]), Ricci tensor (“Ricci” or “contracted curvature collineations” $\mathcal{L}_\xi R_{abc}^c(g) = 0$, cf. [16]). Another generalization is the concept of conformal Killing vector, defined by:

$$\mathcal{L}_\xi g_{ab} = \lambda(x^1, \dots, x^n) g_{ab}, \quad .\tag{20}$$

A subcase are *homothetic* motions with $\lambda = \lambda_0 = \text{const.}$ Conformal Killing vectors are included in what follows. Thus, (15) and (20) are particular subcases of Lie-dragging: they constitute a fixed point in the map of symmetric differentiable tensor fields g_{ab} of full rank defined by Lie-dragging.

4 Weak Lie motions (weak symmetries)

In the 80s, a concept of “p-invariance” has been introduced [17]:

$$\mathcal{L}_\xi \dots \mathcal{L}_\xi g_{ab} = 0, \tag{21}$$

with p Lie derivatives, $p > 1$, acting on the metric. At the time, for $p = 2$ an application has been given in Einstein-Maxwell theory [18]. In the following we will concentrate on this case $p = 2$.

Definition 1:

An infinitesimal point transformation $x \rightarrow x + \xi$ satisfying

$$\mathcal{L}_\xi \mathcal{L}_\xi g_{ab} = 0, \quad \mathcal{L}_\xi g_{ab} \neq 0, \quad (22)$$

generates a “weak Lie motion”.

A coordinate-free formulation of (22) is:

$$\mathcal{L}_W \mathcal{L}_Z g(X, Y) = [W, Z]g(X, Y) - g([W, [Z, X]], Y) - g(X, [Y, [W, Z]]).$$

If applied to other geometric objects, we call (22) “weak symmetry”.¹¹ We also use the expression *weak isometry*.

(22) can be read in two ways:

- The metric g_{ab} is given; determine the generator ξ of a weak Lie motion;
- A vector field or a Lie algebra is given; determine the metric g_{ab} which allows these fields as weak Lie motions.

As has been pointed out in [17], a disadvantage of the new concept is that $\mathcal{L}_\xi \mathcal{L}_\xi g^{ab} = 0$ does not follow from $\mathcal{L}_\xi \mathcal{L}_\xi g_{ab} = 0$ for $\mathcal{L}_\xi g_{ab} \neq 0$. In fact:

$$\mathcal{L}_\xi \mathcal{L}_\xi g^{ab} = -g^{as} g^{bt} \mathcal{L}_\xi \mathcal{L}_\xi g_{st} + 2g^{at} g^{bp} g^{sq} (\mathcal{L}_\xi g_{pq}) (\mathcal{L}_\xi g_{st}). \quad (23)$$

Consequently, in general $\mathcal{L}_\xi \mathcal{L}_\xi g^{ab} = 0$ and $\mathcal{L}_\xi \mathcal{L}_\xi g_{ab} = 0$ define slightly different invariance concepts. If both conditions are imposed, $\mathcal{L}_\xi g_{ab} = \Phi(x) k_a k_b$ with the null vector k_a ($g^{rs} k_r k_s = 0$), and arbitrary scalar function Φ follows. In this case, we call the weak motion generated by $X = \xi^a \frac{\partial}{\partial x^a}$ a *super weak motion*. It entails the existence of a null vector k_a with $\mathcal{L}_\xi k^a = -k^a \mathcal{L}_\xi (\ln \Phi)$.¹² In Euclidean space $\mathcal{L}_\xi g_{ab} = 0$ results. For $p > 2$ the situation would become still more complicated.

¹¹In the set of solutions of (22), the isometries (motions) must also occur. We speak of *genuine* weak Lie motions when motions are to be excluded.

¹²In general relativity, $T^{ab} = \Phi(x) k_a k_b$ describes a null-fluid. What is called here super-weak motion, would have been named *cosymmetric-2-invariance* in ([17], p. 138).

4.1 First examples and generalizations

4.1.1 Weak symmetries

That a weak symmetry can be really weaker than a symmetry is seen already when the Lie derivative is applied twice to a function $f(x^1, \dots, x^n)$:

$$\mathcal{L}_X \mathcal{L}_X f = \mathcal{L}_\xi \mathcal{L}_\xi f = XXf \stackrel{!}{=} 0. \quad (24)$$

In n -dimensional Euclidean space R^n , for a translation in the direction of the k -axis with $\xi^i = \delta_{(k)}^i$, we obtain from (24) $f = x^k f_1(x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^n) + f_2((x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^n)$ in place of $f = f((x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^n)$ for $\mathcal{L}_\xi f \stackrel{!}{=} 0$. For the full translation group of R^n , (24) leads to a polynomial of degree n in the variables (x^1, \dots, x^n) with constant coefficients and linear in each variable (x^1, \dots, x^n) . Thus, for $n = 3$, $f = c_{123}x^1x^2x^3 + \sum_{r,s=1; r < s}^3 c_{rs}x^rx^s + \sum_{s=1}^3 c_sx^s + c_0$ as compared to $f = f_0$ for the translation group as a group of motions.¹³

For a rotation $R_k^i = x^i \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^i}$ (i, k fixed), a function satisfying $\mathcal{L}_\xi \mathcal{L}_\xi f \stackrel{!}{=} 0$ is given by $f = \alpha_1(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{k-1}, x^{k+1}, \dots, x^n) \times \arctan \frac{x^i}{x^k} + \alpha_2(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{k-1}, x^{k+1}, \dots, x^n)$, with $\mathcal{L}_\xi f = -\alpha_1 \neq 0$ for this rotation. For the full rotation group $SO(3)$ in 3-dimensional space, $f = f(\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2})$ follows: no genuine weak motion is possible in this case. These examples show that the set of weak-Lie invariant *functions* can be larger.

A *generalization* of a subgroup of the abelian translation group in an n -dimensional euclidean space is given by:

$$x^{1'} = x^1 + G^1(x^{k+1}, \dots, x^n), \dots, x^{k'} = x^k + G^k(x^{k+1}, \dots, x^n), x^{(k+1)'} = x^{k+1}, \dots, x^{n'} = x^n, \quad (25)$$

with arbitrary C^∞ functions G^1, G^2, \dots, G^k . Weak Lie symmetry under this group for the function $f(x^1, \dots, x^n)$ leads to the same result as for the translation group, although (25) no longer is a Lie group

A link between weak Lie symmetry of scalars and weak Lie motions can be found in conformally flat metrics: $g_{ab} = f(x^1, x^2, \dots, x^n) \eta_{ab}$ due to

$$\mathcal{L}_\xi \mathcal{L}_\xi g_{ab} = (\mathcal{L}_\xi \mathcal{L}_\xi f) \eta_{ab} + 2\mathcal{L}_\xi f \mathcal{L}_\xi \eta_{ab} + \mathcal{L}_\xi \mathcal{L}_\xi \eta_{ab} \quad (26)$$

¹³Note that this result follows only if definition 3 for a complete set of weak symmetries is applied, cf. next section.

In the special case of (20) follows:

$$\mathcal{L}_\xi \mathcal{L}_\xi g_{ab} = (\lambda^2 + \lambda_{,s} \xi^s) g_{ab} , \quad \mathcal{L}_\xi \mathcal{L}_\xi g^{ab} = (\lambda^2 - \lambda_{,s} \xi^s) g^{ab} . \quad (27)$$

Hence, in this case nothing new is obtained by letting the Lie-derivative act twice. The concept of conformal Killing vector could also be weakend to *weak conformal Killing vector* by the demand:

$$\mathcal{L}_\xi \mathcal{L}_\xi g_{ab} = \lambda(x^i) g_{ab} , \quad \mathcal{L}_\xi g_{ab} \neq \mu(x^j) g_{ab} . \quad (28)$$

4.1.2 Weak collineations

For *weak Lie affine collineations*, we find:

$$\begin{aligned} \mathcal{L}_\xi \mathcal{L}_\xi \Gamma_{ab}{}^c(g) &= \xi^s \overset{g}{\nabla}_{(a} [\mathcal{L}_\xi \Gamma_{b)s}{}^c(g)] + [\mathcal{L}_\xi \Gamma_{bs}{}^c(g)] \overset{g}{\nabla}_a \xi^s \\ &\quad + [\mathcal{L}_\xi \Gamma_{as}{}^c(g)] \overset{g}{\nabla}_b \xi^s - [\mathcal{L}_\xi \Gamma_{ab}{}^s(g)] \overset{g}{\nabla}_s \xi^c . \end{aligned} \quad (29)$$

Insertion of $\mathcal{L}_\xi \Gamma_{ab}{}^c(g) = \overset{g}{\nabla}_a \overset{g}{\nabla}_b \xi^c + R^c{}_{bda}(g) \xi^d$ into (29) leads to:

$$\begin{aligned} \mathcal{L}_\xi \mathcal{L}_\xi \Gamma_{ab}{}^c(g) &= \xi^s \overset{g}{\nabla}_{(a} \overset{g}{\nabla}_{b)} \overset{g}{\nabla}_s \xi^c + \xi^s [\overset{g}{\nabla}_{(a} R^c{}_{|ds|b)}] \xi^d + \xi^s R^c{}_{|ds|(b} \overset{g}{\nabla}_{a)} \xi^d \\ &\quad + R^c{}_{dbs} \xi^d \overset{g}{\nabla}_a \xi^s + R^c{}_{das} \xi^d \overset{g}{\nabla}_b \xi^s - R^s{}_{dab} \xi^d \overset{g}{\nabla}_s \xi^c + \overset{g}{\nabla}_b \overset{g}{\nabla}_s \xi^c \overset{g}{\nabla}_a \xi^s \\ &\quad + \overset{g}{\nabla}_a \overset{g}{\nabla}_s \xi^c \overset{g}{\nabla}_b \xi^s - \overset{g}{\nabla}_a \overset{g}{\nabla}_b \xi^s \overset{g}{\nabla}_s \xi^c . \end{aligned} \quad (30)$$

In Minkowski space, the condition is obtained:

$$\xi^s \partial_a \partial_b \partial_s \xi^c + \partial_b \partial_s \xi^c \partial_a \xi^s + \partial_a \partial_s \xi^c \partial_b \xi^s - \partial_a \partial_b \xi^s \partial_s \xi^c = 0 . \quad (31)$$

to be satisfied by the generators of the weak Lie affine collineation. A particular solution is given by $\xi^c = \beta^c f(\alpha_{rs} x^r x^s)$ with constants α_{rs}, β^c and $\beta^s \alpha_{sa} = 0$ and arbitrary C^3 -function f .

If spaces with a Riemannian (Lorentzian) metric are considered, the following expression for weak affine collineations obtains:

$$\mathcal{L}_\xi \mathcal{L}_\xi \{^c_{ab}\} = -g^{cp} g^{sq} \gamma_{pq} [\overset{g}{\nabla}_{(a} \gamma_{b)s} - \frac{1}{2} \overset{g}{\nabla}_s \mathcal{L}_\xi \gamma_{ab}] + g^{cs} [\overset{g}{\nabla}_{(a} \mathcal{L}_\xi \gamma_{b)s} - \frac{1}{2} \overset{g}{\nabla}_s \gamma_{ab} - \mathcal{L}_\xi \{^t_{ab}\} \gamma_{st}] , \quad (32)$$

where γ_{ab} was defined in (10). The concept of weak Lie curvature collineations could also be introduced: $\mathcal{L}_\xi \mathcal{L}_\xi R^c{}_{dab}(g) = 0$. This concept leads to 4th-order PDEs.

4.2 Complete sets of weak Lie motions

If g_{ab} allows the maximal group of motions with $\binom{n+1}{2}$ parameters, no genuine weak Lie motions do exist. If g_{ab} allows a r -parameter group of motions, then $\binom{n+1}{2} - r$ genuine weak Lie motions may exist. The case of a Lie group with $\binom{n+1}{2} - 1$ parameters acting as an isometry group cannot occur in n -dimensional space (Fubini 1903). Hence, in space-time which allows a 10-parameter group as maximal group, no 9-parameter Lie group exists. For 4-dimensional Lorentz-space (with signature ± 2), 8-parameter Lie groups are likewise excluded as isometry groups (Jegorov 1955) ([13], p. 134).¹⁴ Thus, besides the maximal group, the largest group of motions in space-time is a 7-parameter group.¹⁵ In this case, the largest group of weak Lie motions would then be a 3-parameter Lie group.

According to (5), a consequence for weak motions is:¹⁶

$$(\mathcal{L}_{\xi_i}\mathcal{L}_{\xi_j} - \mathcal{L}_{\xi_j}\mathcal{L}_{\xi_i})g_{ab} = \mathcal{L}_{(\mathcal{L}_{\xi_i}\xi_j)}g_{ab} = \mathcal{L}_{c_{ji}^k\xi_k}g_{ab} = c_{ji}^k\mathcal{L}_{\xi_k}g_{ab}. \quad (33)$$

(33) provides a hint about how a *group of weak Lie symmetries* is to be defined when a set of vector fields, ξ, η, ζ, \dots has been found satisfying (22). For genuine weak motions, not all of the following equations can be satisfied: $\mathcal{L}_\eta\mathcal{L}_\xi g_{ab} = 0$, $\mathcal{L}_\xi\mathcal{L}_\eta g_{ab} = 0$, $\mathcal{L}_\eta\mathcal{L}_\zeta g_{ab} = 0$, $\mathcal{L}_\zeta\mathcal{L}_\eta g_{ab} = 0$, $\mathcal{L}_\zeta\mathcal{L}_\xi g_{ab} = 0$, $\mathcal{L}_\xi\mathcal{L}_\zeta g_{ab} = 0$, If the r vectors $\xi_{(k)}$, $k = 1, 2, \dots, r$ are the infinitesimal generators of a Lie group, the above demand *in general* leads into an impasse: instead of its intended role as a weak Lie-invariance group, it reduces to an isometry group. This is due to (5) or (33). An exception holds if some of the vector fields commute.

Consequently, the following definition may be introduced:

Definition 2 (strong complete set):

A Lie algebra presents a strong complete set of weak Lie symmetries if at least one of the corresponding Lie algebra elements does not generate a motion ($\mathcal{L}_{\xi_{(j)}}g_{ab} \neq 0$ for one (j) , at least) and the following $\binom{m+1}{2}$, $m > 1$ conditions hold:

$$\mathcal{L}_{\xi_{(i)}}\mathcal{L}_{\xi_{(j)}}g_{ab} = 0, \quad (34)$$

¹⁴This does not hold for Finsler geometry by which an 8-parameter Lie groups is admitted. Cf. [19], [20], [21]

¹⁵Petrov's claim that for 4-dimensional Lorentz spaces 7-parameter Lie groups are excluded, is not correct, cf. [13], p. 134), [31], p. 122).

¹⁶If an extended Lie algebra is used, on the r.h.s. of (33), the term $2c_{ji}^k{}_{(a}g_{b)c}\xi_k^c$ must be added.

for $(i) = (j)$ and $(i) < (j)$, $(i), (j) = 1, 2, \dots, m$ or, for $(i) = (j)$ and $(i) > (j)$, $(i), (j) = 1, 2, \dots, m$.

The remaining $\mathcal{L}_{\xi_{(i)}} \mathcal{L}_{\xi_{(j)}} g_{ab} \neq 0$ for $(i) > (j)$ [$(i) < (j)$] are then determined through (5). In general, we will demand that none of the vector fields $X_{(i)}$ generate motions.

A less demanding definition would be:

Definition 3 (complete set):

A Lie algebra leads to a complete set of weak Lie-symmetries if each of its infinitesimal operators $X_i = \xi_{(i)}^a \frac{\partial}{\partial x^a}$ generates a weak Lie motion: $\mathcal{L}_{\xi_{(i)}} \mathcal{L}_{\xi_{(i)}} g_{ab} = 0$, $\mathcal{L}_{\xi_{(i)}} g_{ab} \neq 0$ for every $i = 1, 2, \dots, m$.

In section 5.3, examples will be given showing that the alternative definitions 2 and 3 for complete sets of weak Lie symmetries lead to different results. In general, we will prefer definition 2.

As will be seen in the next section, a consequence is that if $g(X, Y)$ allows the *maximal* group of motions, weak Lie motions for $g(X, Y)$ do not exist or reduce to conformal motions. As an example: in 2-dimensional Euclidean space with a 3-parameter maximal group (two translations and one rotation), no genuine weak (Lie) motion exists. The other extremal case is the non-existence of genuine weak Lie motions, e.g., for the rotation group together with definition 2. The Kasner metric (11) which allows three space translations as isometries, is a candidate for not leading to genuine weak Lie motions.

5 Weak Lie invariance

We now want to determine the metrics allowing a time translation and the rotation group as weak Lie motions. The group is chosen such that, as an isometry group, it describes *static, spherically symmetric (s.s.s.) metrics*. Thus we have to allow for four vector fields $\xi_{(i)}$, $i = 1, 2, 3, 4$ forming a Lie algebra with a 2-parameter abelian subalgebra and then drag twice the arbitrary metric g_{ab} . At first, definition 3 is applied and the target metric γ_{ab} calculated.

5.1 Weakly static metrics.

To begin, we demand that only the time translation $T = X_1$ with components $\xi_{(1)}^s = \delta_0^s$ generates a weak motion: $\mathcal{L}_{X_1}\mathcal{L}_{X_1}g_{ab} = 0$. The resulting class of metrics is:

$$g_{ab} = x^0 c_{ab}(x^1, x^2, x^3) + d_{ab}(x^1, x^2, x^3) , \quad (35)$$

with arbitrary symmetric tensors c_{ab}, d_{ab} . The class remains invariant with regard to linear transformations in time $x^0 \rightarrow \alpha(x^1, x^2, x^3)x^0 + \beta(x^1, x^2, x^3)$; α, β arbitrary functions.

5.2 Weak spherical symmetry

Now, the three generators of spatial rotations $SO(3)$ in a representation using polar coordinates $x^1 = r, x^2 = \theta, x^3 = \phi$ are added. Its corresponding generators are:

$$\xi_{(2)}^s = \delta_3^s, \xi_{(3)}^s = -\sin x^3 \delta_2^s - \cos x^3 \text{ctg} x^2 \delta_3^s, \xi_{(4)}^s = \cos x^3 \delta_2^s - \sin x^3 \text{ctg} x^2 \delta_3^s . \quad (36)$$

Lie-dragging with the time translation and with $\xi_{(2)}$ forming the abelian subgroup leads to $\overset{1}{\gamma}_{ab} = g_{ab,0}$, $\overset{2}{\gamma}_{ab} = g_{ab,3}$, and to the weakly Lie invariant metric (i.e., with $\mathcal{L}_{X_1}\mathcal{L}_{X_1}g_{ab} = 0$, $\mathcal{L}_{X_2}\mathcal{L}_{X_2}g_{ab} = 0$)

$$g_{ab} = x^0 x^3 c_{ab}(x^1, x^2) + x^0 d_{ab}(x^1, x^2) + x^3 e_{ab}(x^1, x^2) + f_{ab}(x^1, x^2) \quad (37)$$

with four arbitrary bilinear forms $c_{ab}, d_{ab}, e_{ab}, f_{ab}$.

Lie-dragging with $\xi_{(3)}$ and $\xi_{(4)}$ applied to any of these bilinear forms results in the following equations (using f_{ab} for the presentation):

$$\overset{3}{\gamma}_{ab} = -\sin x^3 f_{ab,2} - 2\cos x^3 f_{2(a}\delta_{b)}^3 + 2\sin x^3 \text{ctg} x^2 f_{3(a}\delta_{b)}^3 + 2\frac{\cos x^3}{\sin^2 x^2} f_{3(a}\delta_{b)}^2 , \quad (38)$$

$$\overset{4}{\gamma}_{ab} = \cos x^3 f_{ab,2} - 2\sin x^3 f_{2(a}\delta_{b)}^3 - 2\cos x^3 \text{ctg} x^2 f_{3(a}\delta_{b)}^3 + 2\frac{\sin x^3}{\sin^2 x^2} f_{3(a}\delta_{b)}^2 . \quad (39)$$

The demand $\overset{2}{\gamma}_{ab} = \overset{3}{\gamma}_{ab} = \overset{4}{\gamma}_{ab} = 0$, i.e., that *spherical symmetry* hold, leads to $c_{ab} = e_{ab} = 0$ and to the well-known result for f_{ab}, d_{ab} :

$$f_{ab} = \alpha(x^1)\delta_a^0\delta_b^0 - \beta(x^1)\delta_a^1\delta_b^1 - \epsilon(x^1)[\delta_a^2\delta_b^2 + \sin^2 x^2 \delta_a^3\delta_b^3] \quad (40)$$

with two free functions $\alpha(x^1), \epsilon(x^1)$.¹⁷

If definition 3 for complete sets of weak symmetry is applied up: two further PDE's must then be satisfied. If all generators of the rotation group are taken into account, then the result is

$$\gamma_{ab} = x^0 d_{ab}(x^1, x^2) + f_{ab}(x^1, x^2) \quad (41)$$

with two bilinear forms d_{ab}, f_{ab} having the same form:

$$f_{ab} = \alpha(x^1) \delta_a^0 \delta_b^0 - \beta(x^1) \delta_a^1 \delta_b^1 - [x^2 \epsilon_1(x^1) + \epsilon_2(x^1)] [\delta_a^2 \delta_b^2 + \sin^2 x^2 \delta_a^3 \delta_b^3] . \quad (42)$$

For the proof, we do not reproduce here the lengthy full expressions for $\mathcal{L}_{\xi_{(3)}} \mathcal{L}_{\xi_{(3)}} f_{ab} = 0$ and $\mathcal{L}_{\xi_{(4)}} \mathcal{L}_{\xi_{(4)}} f_{ab} = 0$, but give only the equations for the components f_{22}, f_{33} :

$$\mathcal{L}_{\xi_{(3)}} \mathcal{L}_{\xi_{(3)}} f_{22} = -\sin^2 x^2 f_{22,2,2} + 2 \frac{\cos^2 x^3}{\sin^2 x^2} [-f_{22} + \frac{f_{33}}{\sin^2 x^2}] \stackrel{!}{=} 0 , \quad (43)$$

$$\mathcal{L}_{\xi_{(4)}} \mathcal{L}_{\xi_{(4)}} f_{22} = -\cos^2 x^2 f_{22,2,2} + 2 \frac{\sin^2 x^3}{\sin^2 x^2} [-f_{22} + \frac{f_{33}}{\sin^2 x^2}] \stackrel{!}{=} 0 . \quad (44)$$

The consequences $f_{22,2,2} = 0$ and $f_{33} = \sin^2 x^2 f_{22}$ are obvious. That (42) is a genuine solution is shown by $\gamma_{22} = \mathcal{L}_{\xi_{(3)}} f_{22} = -\sin x^3 \epsilon_1(x^1) \neq 0$ and by $\gamma_{33} = \mathcal{L}_{\xi_{(3)}} f_{33} = -\sin x^3 \sin^2 x^2 \epsilon_1(x^1) \neq 0$ if $\epsilon_1(x^1) \neq 0$.

The surface $x^1 = \text{const}, x^0 = \text{const}$ has Gaussian curvature:

$$K = \frac{1}{2(\epsilon_1 x^2 + \epsilon_2)^2} [-\epsilon_1 \cot g x^2 + 2\epsilon_1 x^2 + 2\epsilon_2 + \frac{(\epsilon_1)^2}{\epsilon_1 x^2 + \epsilon_2}] . \quad (45)$$

ϵ_1, ϵ_2 are now constants. For $\epsilon_1 \rightarrow 0$ we obtain the constant curvature of the 2-sphere.

The time translation and the 3 generators of the rotation group form a complete set of weak Lie motions; this shows that definition 3 is not empty.

However, if it is asked that the rotation group generate a *strong* set of weak symmetries according to definition 2, then the result is very restrictive. The conditions $\mathcal{L}_{\xi_{(2)}} \mathcal{L}_{\xi_{(3)}} f_{ab} = 0 = \mathcal{L}_{\xi_{(2)}} \mathcal{L}_{\xi_{(4)}} f_{ab}$ for equations (37), (42) are

¹⁷One of the functions $\alpha(x^1), \beta(x^1)$ is superfluous because, locally, a 2-dimensional space is conformally flat. $f_{01} = f_{23} = 0$ follows from the rotation group acting on a 2-dimensional subspace. In addition, here $f_{02} = f_{03} = f_{12} = f_{13} = 0$ has been used.

leading to the remaining metric tensor of (41). If $\mathcal{L}_{\xi_{(3)}}\mathcal{L}_{\xi_{(4)}}\gamma_{33} = 0$ is studied for f_{ab} , then $\mathcal{L}_{\xi_{(3)}}\mathcal{L}_{\xi_{(4)}}f_{ab} \neq 0$ due to the only nonvanishing expression $\mathcal{L}_{\xi_{(3)}}\mathcal{L}_{\xi_{(4)}}f_{33} = \sin x^2 \cos x^2 \times \epsilon_1(x^1)$ for $\epsilon_1(x^1) \neq 0$. Thus the demand that the rotation group in 3 dimensions generates a *strong* set of weak Lie symmetries according to definition 2 enforces $\epsilon_1(x^1) = 0$ and reduces to an isometry. Nevertheless, the resulting spherically symmetric metric is only weakly static.

5.3 The group G_3 acting as a group of weak Lie motions

In taking up the example of a G_3 acting on V_3 from section 2 with Lie algebra (18), we first apply definition 3 to a scalar $f(x^1, x^2, x^3)$. If the generators are to lead to motions, then the only solution is $f = \text{constant}$. Definition 3 for a complete set of weak Lie motions leads to:¹⁸

$$f = a_0 x^2 x^3 + b_0 x^1 (x^2 - x^1 x^3) + c_0 x^1 x^3 + b_1 x^2 + c_1 x^3 + d_1 x^1 + d_0, \quad (46)$$

while definition 2 results in:

$$f = c_0 (x^1 x^3 + x^2) + c_1 x^3 + d_1 x^1 + d_0. \quad (47)$$

We note, that the only one of the 9 possible demands so far unused, i.e., $\mathcal{L}_{\xi_{(3)}}\mathcal{L}_{\xi_{(2)}}g_{ab} = 0$ reduces (47) to

$$f = c_1 x^3 + d_1 x^1 + d_0. \quad (48)$$

Applying G_3 to the metric, the following weakly Lie-invariant metric is obtained:

$$\gamma_{ab} = x^2 \begin{pmatrix} \alpha_{11}^{(0)} & \alpha_{12}^{(0)} & P_1 \\ \alpha_{21}^{(0)} & \alpha_{22}^{(0)} & P'_1 \\ P_1 & P'_1 & P_2 \end{pmatrix} + x^3 \left[x^1 \begin{pmatrix} \alpha_{11}^{(0)} & \alpha_{12}^{(0)} & P_1 \\ \alpha_{21}^{(0)} & \alpha_{22}^{(0)} & P'_1 \\ P_1 & P'_1 & P_2 \end{pmatrix} + \begin{pmatrix} \beta_{11}^{(0)} & \beta_{12}^{(0)} & \tilde{P}_1 \\ \beta_{21}^{(0)} & \beta_{22}^{(0)} & \tilde{P}'_1 \\ \tilde{P}_1 & \tilde{P}'_1 & \tilde{P}_2 \end{pmatrix} \right] + \begin{pmatrix} Q_1 & Q_1 & Q_2 \\ Q_1 & \tilde{Q}_1 & Q_2 \\ Q_2 & Q'_2 & Q_3 \end{pmatrix},$$

where $P_i, P'_i, Q_i, \tilde{Q}_i, Q_i$ are polynomials in the coordinate x^1 of order i , the coefficients of which are not all independent:

¹⁸The calculations are sketched in appendix 2.

$$\begin{aligned}
P_1 &= \overset{(0)}{\alpha}_{12}x^1 + c_{13}, P'_1 = \overset{(0)}{\alpha}_{22}x^1 + c_{23}, P_2 = \overset{(0)}{\alpha}_{22}(x^1)^2 + 2c_{23}x^1 + c_{33}, \\
Q_1 &= \overset{(0)}{l}_{11}x^1 + \overset{(0)}{m}_{11}, Q_1 = \overset{(0)}{l}_{12}x^1 + \overset{(0)}{m}_{12}, \tilde{Q}_1 = \overset{(0)}{l}_{22}x^1 + \overset{(0)}{m}_{22}, \\
Q_2 &= \overset{(0)}{l}_{12}(x^1)^2 + \overset{(0)}{m}_{13}x^1 + \overset{(0)}{k}_{13}, Q_2 = \overset{(0)}{l}_{22}(x^1)^2 + \overset{(0)}{m}_{23}x^1 + \overset{(0)}{k}_{23}, \\
Q_3 &= \overset{(0)}{l}_{22}(x^1)^3 + \overset{(0)}{m}_{23}(x^1)^2 + \overset{(0)}{k}_{33}x^1 + m_{33} \text{ and } \overset{(0)}{\alpha}_{ab}, (a, b = 1, 2), c_{ij}, \overset{(0)}{l}_{ij}, \overset{(0)}{m}_{ij}
\end{aligned}$$

and $\overset{(0)}{k}_{ij}$ constants. In the polynomials $\tilde{P}_1, \tilde{P}'_1, \tilde{P}_2$, the constants α_{ab}, c_{ab} are exchanged by the set of independent constants β_{ab}, d_{ab} . Thus definition 2 is not empty. Two independent matrices of the type that occurred for the group acting as an isometry group and a third, new matrix occur now.

Definition 3 leads to a different complete set of weak Lie motions for which the metric takes the form: $g_{ab} = d_{ab}(x^1)x^2 + e_{ab}(x^1)x^3 + \epsilon_{ab}(x^1)$ with d, e, ϵ expressed by matrices of the form:

$$\begin{pmatrix} P_{11} & P_{12} & Q_1 \\ P_{12} & P_{22} & Q_2 \\ Q_1 & Q_2 & M \end{pmatrix},$$

where P_{ik} are polynomials of 1st degree, Q_i of 2nd degree, and M a polynomial of 3rd degree.

6 A new algebra structure

For Lie-dragging, up to now we have mostly taken vector fields forming Lie algebras corresponding to Lie groups of point transformations. In the following, after an introductory section, we consider more general types both of groups and algebras in sections 7 and 8.

6.1 Lie-dragging for vector fields not forming Lie algebras

Already in (25) of section 4.1.1, vector fields containing free functions were considered. We now continue with vector fields $X_1 = \xi^r \frac{\partial}{\partial x^r}$; $X_2 = \eta^s \frac{\partial}{\partial x^s}$ with $\xi^r = f(x^0)\delta_1^r, \eta^s = h(x^1)\delta_0^s$ such that

$$[X_1, X_2] = f(x^0)H(x^1)X_2 - h(x^1)F(x^0)X_1. \quad (49)$$

Here, $F(x^0) = \frac{d(\ln f(x^0))}{dx^0}$, $H(x^1) = \frac{d(\ln h(x^1))}{dx^1}$. The finite transformations belonging to X_1 and X_2 , respectively, are generalized time- and space-translations

$$x^0 \rightarrow x^{0'} = x^0 + h(x^1); \quad x^1 \rightarrow x^{1'} = x^1 + f(x^0) \quad (50)$$

leaving invariant the time interval $|x_{(i)}^0 - x_{(j)}^0|$ and the space interval $|x_{(i)}^1 - x_{(j)}^1|$ between two events $(x_{(i)}^0, x_{(i)}^1)$ and $(x_{(j)}^0, x_{(j)}^1)$. Each of the transformations

$$x^0 \rightarrow x^{0'} = x^0 + h(x^1); \quad x^1 \rightarrow x^{1'} = x^1 + a, \quad (51)$$

and

$$x^1 \rightarrow x^{1'} = x^1 + f(x^0); \quad x^0 \rightarrow x^{0'} = x^0 + b \quad (52)$$

forms a group: $x^{0''} = x^{0'} + k(x^{1'}) = x^0 + h(x^1) + k(x^1 + a)$, $x^{1''} = x^1 + A + a$, and $x^{1''} = x^{1'} + g(x^{0'}) = x^1 + f(x^0) + g(x^0 + b)$, $x^{0''} = x^0 + B + b$. However, these groups are *not* Lie groups: in part, the Lie-group parameters have been replaced by arbitrary functions. In this case, the algebra (49) reduces to either

$$[X_1, X_2] = H(x^1)X_2. \quad (53)$$

or to

$$[X_1, X_2] = -F(x^0)X_1. \quad (54)$$

Likewise, (49), (53), and (54) are not Lie algebras.

Both transformations (50) applied together: $x^{0''} = x^{0'} + k(x^{1'}) = x^0 + h(x^1) + k(x^1 + f(x^0))$, $x^{1''} = x^{1'} + g(x^{0'}) = x^1 + f(x^0) + g(x^0 + h(x^1))$ do not even form a group.

The class of functions involved may be narrowed considerably by the demand that the function f of a special type be kept fixed, e.g., be a polynomial of degree p , or $f(x^0) = a \sin x^0 + b \cos x^0$. In these cases, just one function with constant coefficients occurs in the group; the group transformations change only the coefficients. (52) is a subgroup of the so-called *Mach-Poincaré group* $G_4(3)$ [23], ([24], pp. 85-101):

$$x^{a'} = A^a_r x^r + f^a(x^0), \quad x^{0'} = x^0 + b, \quad A^a_r A^r_b = \delta^a_b. \quad (55)$$

This group plays a role in Galilean relative mechanics.

A generalization is the group $G_1(6)$ of transformations leaving invariant the observables describing a *rigid body*; 6 free functions of x^0 and one Lie-group parameter do appear:

$$x^{i'} = A^i_j(x^0)x^j + f^i(x^0), \quad x^{0'} = x^0 + b, \quad A^i_j(x^0)A^j_k(x^0) = \delta^i_k, \quad (i, j = 1, 2, 3). \quad (56)$$

The corresponding seven algebra generators are:

$$\begin{aligned}
T &= \frac{\partial}{\partial x^0}, \quad X_i = f_i(x^0) \frac{\partial}{\partial x^i} \quad (i \text{ not summed}), \\
Y_1 &= \omega_3^2(x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}), \quad Y_2 = \omega_3^1(x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3}), \quad Y_3 = \omega_2^1(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2})
\end{aligned} \tag{57}$$

with $\omega_j^i = \omega_j^i(x^0)$. The corresponding algebra is given by:

$$\begin{aligned}
[T, T] &= 0, \quad [T, X_i] = F_i(x^0) X_i, \quad F_i = \frac{d}{dx^0} \ln(f_i(x^0)), \quad [X_i, X_j] = 0, \quad (i, j = 1, 2, 3) \\
[T, Y_1] &= \omega_3^2(x^0) Y_1, \quad [T, Y_2] = \omega_3^1(x^0) Y_2, \quad [T, Y_3] = \omega_2^1(x^0) Y_3, \quad \omega_j^i = \frac{d}{dx^0} \ln(\omega_j^i(x^0)), \\
[Y_1, Y_2] &= -\frac{\omega_3^1 \omega_3^2}{\omega_2^1} Y_3, \quad [Y_2, Y_3] = -\frac{\omega_2^1 \omega_3^1}{\omega_3^2} Y_1, \quad [Y_1, Y_3] = -\frac{\omega_2^1 \omega_2^3}{\omega_3^1} Y_2, \\
[X_1, Y_1] &= 0, \quad [X_1, Y_2] = \frac{f_1(x^0)}{f_3(x^0)} \omega_3^1 X_3, \quad [X_1, Y_3] = -\frac{f_1(x^0)}{f_2(x^0)} \omega_2^1 X_2, \\
[X_2, Y_1] &= \frac{f_2(x^0)}{f_3(x^0)} \omega_3^2 X_3, \quad [X_2, Y_2] = 0, \quad [X_2, Y_3] = \frac{f_2(x^0)}{f_1(x^0)} \omega_2^1 X_1, \\
[X_3, Y_1] &= \frac{f_3(x^0)}{f_2(x^0)} \omega_3^2 X_2, \quad [X_3, Y_2] = -\frac{f_3(x^0)}{f_1(x^0)} \omega_3^1 X_1, \quad [X_3, Y_3] = 0.
\end{aligned} \tag{58}$$

There exist further groups of this non-Lie type occurring in classical mechanics like Weyl's kinematical group $G_3(6)$ and the covariance group of the Hamilton-Jacobi equation $G_7(3)$ or, as a subgroup in non-relativistic quantum mechanics, the covariance group of the Schrödinger equation $G_{12}(0)$, cf. [24]. The structure functions of all these groups depend on a single coordinate, the time.

7 Extended Lie Algebras

In the following, we will deal with a subbundle of the tangent bundle of n -dimensional Euclidean or Lorentz space. We will permit that the structure constants in the defining relations for a Lie algebra become dependent on the components ξ_i^a of the vector fields $X_i(x)$: they will become *structure functions*.

Definition 4:

The algebra

$$[X_i, X_j] = c_{ij}^k(x^1, x^2, \dots, r)X_k \quad (59)$$

with structure functions $c_{ij}^k(x^1, x^2, \dots, x^r)$ is called an *extended Lie algebra*.

The Lie algebra elements form an “involutive distribution”. This is “a smooth distribution V on a smooth manifold M , i.e., a smooth vector subbundle of the tangent bundle” TM . The Lie brackets constitute the composition law; the injection $V \hookrightarrow TM$ functions as the anchor map (cf. [25], p. 13). This is a simple example for a *tangent Lie algebroid* (cf. also ([5], p. 100 and example 2.7, p. 105)).¹⁹ Nevertheless, the involutive distribution used here can also be considered a subset of the infinite-dimensional “Lie-algebra” $\mathcal{B}(M)$ of footnote 5.

After completion of the paper, I learned of some of the historical background of (59): It already has occurred as the condition for closure of a complete set of linear, homogeneous operators belonging to a complete system of 1st order PDE’s in Jacobi’s famous paper of 1862 ([29], §26, p. 40).²⁰

(7) must then be replaced by

$$\mathcal{L}_{X_i}\mathcal{L}_{X_j}X_k = (c_{jk}^l c_{il}^m + X_i c_{jk}^m)X_m, \quad (60)$$

and (8) by

$$c_{jk}^l c_{il}^m + c_{ij}^l c_{kl}^m + c_{ki}^l c_{jl}^m + X_i c_{jk}^m + X_k c_{ij}^m + X_j c_{ki}^m = 0. \quad (61)$$

An *extended Cartan-Killing* form can be defined acting as a symmetric metric on the sections of the subtangent bundle. An asymmetric form could be defined as well.

Definition 5 (Generalized Cartan-Killing form):

The generalized Cartan-Killing bilinear form τ is defined by:

$$\tau_{ij} := \sigma_{ij} + 2X_{(i}c_{j)m}^m = c_{il}^m c_{jm}^l + 2X_{(i}c_{j)m}^m. \quad (62)$$

¹⁹Closely related, but different structures are *family of Lie algebras* [26], [27] and *variable Lie algebras* ([28], p. 115).

²⁰In Jacobi’s paper, (59) is used in phase space such that the structure functions depend on both coordinates and momenta: $c_{ij}^k(x^1, x^2, \dots, x^r, p_1, p_2, \dots, p_r)$. It is in Clebsch’s paper of 1866 ([30], §1) in connection with his definition of a complete system of linear PDE’s that the r.h.s. of (59) depends only on the coordinates. Cf. also equation (3.1) in [3], p. 311.

The generalized Cartan-Killing form now depends on the base points of the fibres in the tangent bundle. They may be interpreted as a metric.

To use the example of the group $G_1(6)$ given in section 6.1: The structure functions for the corresponding extended algebra (58) of rigid body transformations are shown in appendix 4. From them, calculation of the extended Cartan-Killing form leads to a Lorentz metric with signature (1,3) of rank 4 within a degenerated 7-dimensional bilinear form:

$$\tau_{ij} = \begin{pmatrix} \tau_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_{66} \end{pmatrix} \quad (63)$$

where $\tau_{00} = \sum_{i=1}^3 \frac{\ddot{f}_i}{f_i} + \frac{\ddot{\omega}_3^2}{\omega_3^2} + \frac{\ddot{\omega}_1^3}{\omega_1^3} + \frac{\ddot{\omega}_2^1}{\omega_2^1}$, and $\tau_{44} = -4(\omega_3^2)^2$, $\tau_{55} = -4(\omega_3^1)^2$, $\tau_{66} = -4(\omega_2^1)^2$. By projection into the 4-dimensional space with coordinates 0, 4, 5, 6 and signature (1,3), we surprisingly arrive at the general class of one-dimensional gravitational fields [31]. For special values for the f_i , and ω_i^k , the Kasner metric (11) can be derived by this approach. All the pre-relativistic groups mentioned at the end of the previous section lead to Cartan-Killing forms depending on just one coordinate, the time.

The following definition introduces a new class of extended motions and a new class of weak extended motions, the infinitesimal generators of which form an extended Lie algebra.

Definition 6 (extended motions):

Let $x \rightarrow x + \xi$, $y \rightarrow y + \eta$ be infinitesimal transformations forming a continuous group the corresponding algebra of which is an extended Lie algebra according to definition 4. Then, the vector fields $X = \xi^c \frac{\partial}{\partial x^c}$, $Y = \eta^c \frac{\partial}{\partial x^c}$ with $\mathcal{L}_X g_{ab} = 0$, $\mathcal{L}_Y g_{ab} = 0$ are called *extended motions*.

An analogous formulation is:

Definition 7 (extended weak motions):

Let $x \rightarrow x + \xi$, $y \rightarrow y + \eta$ be infinitesimal transformations forming a con-

tinuous group the corresponding algebra of which is an extended Lie algebra according to definition 4. Then, the vector fields $X = \xi^c \frac{\partial}{\partial x^c}$, $Y = \eta^c \frac{\partial}{\partial x^c}$ with $\mathcal{L}_X \mathcal{L}_X g_{ab} = 0$, $\mathcal{L}_Y \mathcal{L}_Y g_{ab} = 0$ are called *extended weak motions*.

8 Extended motions and extended weak (Lie) motions

In section (6.1), we have given examples of non-Lie groups leading to extended Lie algebras. How will the corresponding extended motions and extended weak (Lie) motions differ? These concepts are exemplified here with the most simple non-Lie group (52). The tangent vectors X , Y with the algebra (54) form an extended motion ($\mathcal{L}_X g_{ab} = 0$, $\mathcal{L}_Y g_{ab} = 0$) for all metrics of maximal rank 3:

$$g_{ab} = \begin{pmatrix} \alpha_{00}(x^2, x^3) & 0 & \alpha_{02}(x^2, x^3) & \alpha_{03}(x^2, x^3) \\ 0 & 0 & 0 & 0 \\ \alpha_{02}(x^2, x^3) & 0 & \alpha_{22}(x^2, x^3) & \alpha_{23}(x^2, x^3) \\ \alpha_{03}(x^2, x^3) & 0 & \alpha_{23}(x^2, x^3) & \alpha_{33}(x^2, x^3) \end{pmatrix}, \quad (64)$$

with arbitrary functions α_{ab} due to arbitrariness of $f(x^0)$. This is to be compared with the motions derived from $X = \frac{\partial}{\partial x^1}$, $Y = \frac{\partial}{\partial x^0}$ forming an abelian Lie algebra and leading to

$$g_{ab} = \alpha_{ab}(x^2, x^3). \quad (65)$$

The corresponding extended weak (Lie) motions ($\mathcal{L}_X \mathcal{L}_X g_{ab} = 0$, $\mathcal{L}_Y \mathcal{L}_Y g_{ab} = 0$) are given by:

$$g_{ab} = \begin{pmatrix} x^1 \alpha_{00} + \beta_{00} & \beta_{01} & x^1 \alpha_{02} + \beta_{02} & x^1 \alpha_{03} + \beta_{03} \\ \beta_{01} & 0 & \beta_{12} & \beta_{13} \\ x^1 \alpha_{02} + \beta_{02} & \beta_{12} & x^1 \alpha_{22} + \beta_{22} & x^1 \alpha_{23} + \beta_{23} \\ x^1 \alpha_{03} + \beta_{03} & \beta_{13} & x^1 \alpha_{23} + \beta_{23} & x^1 \alpha_{33} + \beta_{33} \end{pmatrix}, \quad (66)$$

where $\alpha_{ab} = \alpha_{ab}(x^2, x^3)$; $\beta_{ab} = \beta_{ab}(x^2, x^3)$, $\alpha_{0a} = 0$, $\beta_{11} = 0$. Comparison with the weak (Lie) motions generated by the translations given above shows the class of metrics:

$$g_{ab} = x^1 \alpha_{ab}(x^2, x^3) + \beta_{ab}(x^2, x^3). \quad (67)$$

9 Two-dimensional extended Lie algebras

In section 6.1 we have given the example (49) showing that (59) is not empty. As for Lie algebras, the question about a classification of extended Lie algebras in n -dimensional space arises. This being a topic of its own, we start here by considering the case $n = 2$ only, without proving completeness of the result.

We begin with:²¹

$$[X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 \quad (68)$$

with $X_1 = \xi^1 \frac{\partial}{\partial x_1} + \xi^2 \frac{\partial}{\partial x_2}$, $X_2 = \eta^1 \frac{\partial}{\partial x_1} + \eta^2 \frac{\partial}{\partial x_2}$. This is a system of two equations for the 6 unknowns ξ^i, η^j and $c_{12}^i, i = 1, 2$:

$$[X_1, X_2] = [\xi^1 \eta^1_{,1} + \xi^2 \eta^1_{,2} - \eta^1 \xi^1_{,1} - \eta^2 \xi^1_{,2}] \frac{\partial}{\partial x_1} + [\xi^1 \eta^2_{,1} + \xi^2 \eta^2_{,2} - \eta^1 \xi^2_{,1} - \eta^2 \xi^2_{,2}] \frac{\partial}{\partial x_2}. \quad (69)$$

We distinguish two cases according to whether the vector fields are unaligned or aligned. In the *first case*, for $\xi^1 \neq 0, \eta^2 \neq 0$:

$$[X_1, X_2] = [(\xi^1)^2 \frac{\partial}{\partial x_1} \left(\frac{\eta^1}{\xi^1} \right) + \xi^2 \eta^1_{,2} - \eta^2 \xi^1_{,2}] \frac{\partial}{\partial x_1} + [\xi^1 \eta^2_{,1} - \eta^1 \xi^2_{,1} - (\eta^2)^2 \frac{\partial}{\partial x_2} \left(\frac{\xi^2}{\eta^2} \right) \frac{\partial}{\partial x_2}]. \quad (70)$$

Here, the simplification $\xi^2 = \eta^1 = 0$ does not restrict generality. In the solution, two free functions ξ^1, η^2 remain; they are contained in the expressions for the structure functions:

$$c_{12}^1 = -\frac{\eta^2}{\xi^1} \xi^1_{,2}, \quad c_{12}^2 = \frac{\xi^1}{\eta^2} \eta^2_{,1}. \quad (71)$$

The further simplification $\xi^1 = \eta^2$ leads to:

$$[X_1, X_2] = -\xi^1_{,2} X_1 + \xi^1_{,1} X_2 \quad (72)$$

with arbitrary $\xi^1 = \xi^1(x^1, x^2)$. Calculation of the extended Cartan-Killing form (62) results in:

$$\tau_{ik} = \begin{pmatrix} ([\xi^1_{,1}]^2) + \xi^1 \xi^1_{,1,1} & \xi^1_{,1} \xi^1_{,2} + \xi^1 \xi^1_{,1,2} \\ \xi^1_{,1} \xi^1_{,2} + \xi^1 \xi^1_{,1,2} & (\xi^1_{,2})^2 + \xi^1 \xi^1_{,2,2} \end{pmatrix},$$

²¹The coordinates x^0, x^1 of section 6.1 are replaced by x^1, x^2 .

or simply

$$\tau_{ij} = \frac{1}{2}[(\xi^1)^2]_{,ij} . \quad (73)$$

In general $\det(\tau_{ik}) \neq 0$.

In order to find (54) in this formalism, we must start from (52) and set $\xi^1 = f(x^0)$, $\eta^0 = 1$ such that $c_{12}^1 = -F(x^0)$, $c_{12}^2 = 0$. As the only dependence is on x^0 , the Cartan-Killing form degenerates (does not have full rank). This also happens for the algebra (58).

For 2-dimensional Lorentz space, one of the generators can be lightlike. We use only the simplification $\xi^2 = 0$ and $\eta^1 = \pm\eta^2$ such that in this case the relation:

$$[X_1, X_2] = \left[\frac{\pm 1}{(\xi^1)^2} \frac{\partial}{\partial x_1} \left(\frac{\eta^1}{\xi^1} \right) - \eta^2 \xi_{,2}^1 \right] \frac{\partial}{\partial x_1} + \xi^1 \eta_{,1}^2 \frac{\partial}{\partial x_2} \quad (74)$$

follows. Again, we can set $\xi^1 = \eta^2$ and come back to (72).

The two different Lie algebras allowed in 2-dimensional space can be obtained from (72) by special choice of ξ^1 . By redefinition of the algebra elements in the sense of

$$X_1 \rightarrow Y_1 = f(x^1, x^2)X_1 + g(x^1, x^2)X_2, \quad X_2 \rightarrow Y_2 = m(x^1, x^2)X_1 + p(x^1, x^2)X_2 \quad (75)$$

with arbitrary functions f, g, m, p , from (68) it may be possible to come back to the canonical form for the non-abelian Lie algebra. However, this is an open question.²²

In the *second case* of aligned tangent vectors we can set $\xi^2 = 0 = \eta^2$. From (69) we retain as the only structure function:

$$c_{12}^1 = \xi^1 \eta_{,1}^1 - \eta^1 \xi_{,1}^1 . \quad (76)$$

The Cartan-Killing form then is:

$$\tau_{ik} = \begin{pmatrix} 0 & -\xi_1(\xi^1 \eta_{,1}^1 - \eta^1 \xi_{,1}^1) \\ -\xi_1(\xi^1 \eta_{,1}^1 - \eta^1 \xi_{,1}^1) & (\xi^1 \eta_{,1}^1 - \eta^1 \xi_{,1}^1)^2 - \eta_1(\xi^1 \eta_{,1,1}^1 - \eta^1 \xi_{,1,1}^1) \end{pmatrix} . \quad (77)$$

²²It depends on whether solutions of certain nonlinear 1st order PDEs exist.

In general, there is a wealth of possibilities available for setting up extended Lie algebras. A particular choice for the structure functions would be²³

$$c_{(i)(j)}^{(k)} := \xi_{(i)}^r \xi_{(j)}^s g_{rs} [\delta_{(i)}^{(k)} - \delta_{(j)}^{(k)}] . \quad (78)$$

In Euclidean space $g_{rs} = \delta_{rs}$, in Lorentz space $g_{rs} = \eta_{rs}$ are most simple choices. It is not difficult to calculate the extended Cartan-Killing form which depends only on the inner products $\xi_{(i)}^r \xi_{(j)}^s g_{rs}, (i), (j) = 1, 2, \dots, m; r, s = 1, 2, \dots, n$ ($0, 1, 2, \dots, n-1$).

10 Discussion and conclusion

When Lie-dragging is seen as a mapping in the space of metrics, it may be asked whether it could provide a method for generating solutions of Einsteins equations from known solutions. It is easily shown that the Schwarzschild vacuum solution, the Robertson-Walker metric with flat 3-spaces, and the Kasner metric cannot be obtained by Lie-dragging of *Minkowski* space. On the other hand, the metric (37) which is weakly Lie-invariant with respect to the group (T, SO(3)) trivially contains cosmological solutions of Einsteins equation. If the metric $x^0 d_{ab}(x^1, x^2)$ with spherically symmetry and with flat space sections is chosen, by a transformation of the time coordinate we arrive at the line element

$$ds^2 = (d\tau)^2 - 2/3\tau^{3/2}[(dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2] . \quad (79)$$

It describes a cosmic substrate with the equation of state $p = -\frac{1}{9}\mu$, where μ describes pressure and μ the energy density of the material. This equation of state for $w = -\frac{1}{9}$ is non-phantom because of $-1 < w$ but does not accelerate the expansion of the universe which occurs for $-1 < w < -\frac{1}{3}$.

It remains to be seen whether the anisotropic line element

$$ds^2 = (d\tau)^2 - c_0\tau^{2/3}[c_1\theta r + c_2][(dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2] \quad (80)$$

²³In (78), the summation convention is used for unbracketed indices.

can satisfy Einstein's equations with a reasonable matter distribution. In view of the fact that Lie-dragging does not preserve the rank of the metric, its efficiency for generating interpretable gravitational fields is reduced considerably.

Surprisingly, by studying the rigid body transformations $G_1(6)$ as a group of extended motions, we arrived at the complete class of one-dimensional gravitational fields including the Kasner metric. More generally, a close relation to *finite* transformation groups in classical, non-relativistic mechanics containing arbitrary functions has been established. It is still to be cleared up whether a connection to gauge theory in physics exists.

A classification of solutions of Einstein's equations with regard to weak (Lie) symmetries could be made. Although this might be a further help for deciding whether two solutions are transformable into each other or not, the calculational effort looks extensive.

Weak Lie-invariance as a weakened concept of "symmetry" has been introduced and its consequences presented through a number of examples. It also has led to the introduction of a new type of algebra ("extended Lie algebra") which is an example for a tangent Lie algebroid. In each *fibre* of a subbundle of the tangent bundle, the "extended Lie algebra" reduces to a Lie algebra. By help of an extended Cartan-Killing form, Riemann or Lorentz metrics have been constructed on such an algebroid. A particular example is provided by the non-Lie groups of classical mechanics mentioned above. The ensuing possible geometries could be studied and classified in the spirit of Felix Klein. A classification of non-Lie groups leading to extended Lie algebras and of the extended Lie algebras could also be of interest. A further study of the concept of extended Lie algebras is needed and might be of some relevance.

Whether there are noteworthy applications in geometry and physics beyond those established here for classical mechanics and the Schrödinger equation will have to be found out.

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12 Appendix

12.1 Appendix 1 (Integrability conditions)

That an *arbitrary* symmetrical tensor field of fixed rank *cannot* be reached by the operation of Lie-dragging may be seen already from (15), (13), or from the following equations obtained from (16):

$$\overset{g}{\nabla}_b \overset{g}{\nabla}_c \xi_a + \xi_d R^d_{bca}(g) = 1/2 [\overset{g}{\nabla}_b \gamma_{ac} + \overset{g}{\nabla}_c \gamma_{ba} + \overset{g}{\nabla}_a \gamma_{cb}] . \quad (81)$$

Here, $R^d_{bca}(g)$ is the curvature tensor of the metric g_{ab} . For g_{ab}, γ_{ab} fixed, the n^3 equations (81) would be an integrability condition for the n components of the vector field ξ . Eq. (81) generalizes part of the integrability condition for (15) given in [22], eq. (6.2), p. 56. As an example, we take Minkowski space $g_{ab} = \eta_{ab}$, a space of maximal symmetry. From (81) with $\gamma_{ab} = 2\xi_{(a,b)}$ follows:

$$\partial_b \partial_c \xi_a = \partial_b \xi_{(a,c)} + \partial_c \xi_{(b,a)} + \partial_a \xi_{(c,b)} , \quad (82)$$

the general solution of which, apart from the generators of the Poincaré group, is $\xi_a = c_0 \eta_{ar} x^r + \partial_r F^{[rs]} \eta_{as}$ (cf. appendix 3). Thus, a homothetic motion appears as well.

If we look at this equation as a condition for γ_{ab} when the vector field ξ and the metric g_{ab} are given, the equation then says that there are linear relations between the first derivatives of γ_{ab} . Further differentiation of (81) leads to:

$$\begin{aligned} \overset{g}{\nabla}_d \overset{g}{\nabla}_b \overset{g}{\nabla}_c \xi_a + \overset{g}{\nabla}_a \overset{g}{\nabla}_b \overset{g}{\nabla}_d \xi_c + \overset{g}{\nabla}_c \overset{g}{\nabla}_b \overset{g}{\nabla}_a \xi_d = 1/2 [\overset{g}{\nabla}_d \overset{g}{\nabla}_b \gamma_{ac} + \overset{g}{\nabla}_a \overset{g}{\nabla}_b \gamma_{cd} + \overset{g}{\nabla}_c \overset{g}{\nabla}_b \gamma_{da}] \\ + \gamma_{as} R^s_{bcd}(g) + \gamma_{cs} R^s_{bda}(g) + \gamma_{ds} R^s_{bac}(g) = 0 . \end{aligned} \quad (83)$$

A counting of derivatives and equations leads to the number of restrictions for the obtainable γ_{ab} showing up explicitly as relations among the derivatives of γ_{ab} .

12.2 Appendix 2

We find

$$\mathcal{L}_{X_1}\mathcal{L}_{X_1} = f_{,2,2} = 0, \quad \mathcal{L}_{X_2}\mathcal{L}_{X_2} = f_{,3,3} = 0, \quad (84)$$

$$\mathcal{L}_{X_3}\mathcal{L}_{X_3} = f_{,1,1} - 2x^3 f_{,2,1} + (x^3)^2 f_{,2,2} = 0, \quad \mathcal{L}_{X_1}\mathcal{L}_{X_2} = \mathcal{L}_{X_2}\mathcal{L}_{X_1} = f_{,2,3} = 0, \quad (85)$$

$$\mathcal{L}_{X_1}\mathcal{L}_{X_3} = -f_{,1,2} + x^3 f_{,2,2} = 0, \quad \mathcal{L}_{X_3}\mathcal{L}_{X_1} = -f_{,1,2} + x^3 f_{,2,2} = 0, \quad (86)$$

$$\mathcal{L}_{X_2}\mathcal{L}_{X_3} = -f_{,1,3} + f_{,2} + x^3 f_{,2,3} = 0, \quad \mathcal{L}_{X_3}\mathcal{L}_{X_2} = -f_{,3,1} + x^3 f_{,3,2} = 0, \quad (87)$$

from which the results (46)-(48) follow.

12.3 Appendix 3

From (82), by contraction with η^{bc} the equation $\partial_a(\partial^c \xi_c) = 0$ follows, whence $\partial^c \xi_c = c_0 = \text{const}$. Contraction with η^{ac} then leads to $(\nabla)^2 \xi_c = 0$. The most general Ansatz for solving $\partial^c \xi_c = c_0$ is $\xi^c = \frac{c_0}{4} x^c + \partial_r F^{[rc]}$ with $\partial_r(\nabla)^2 F^{[rc]} = 0$. Let $X^c := \partial_r F^{[rc]}$; then, from (82) $\partial_c(\partial_a X_b + \partial_b X_a) = 0$. Whence follow the equation for a homothetic motion.

12.4 Appendix 4

If we use the notation $Z_1 = T, Z_i = X_i, (i = 1, 2, 3), Z_j = Y_j (j = 1, 2, 3)$ where X_i correspond to time-dependent translations, Y_j to time-dependent rotations, the structure functions are given by:

$$\begin{aligned} c_{0i}{}^i &= (\ln f_i) \dot{} (i = 1, 2, 3), c_{04}{}^4 = (\ln \omega_3^2) \dot{}, c_{05}{}^5 = (\ln \omega_3^1) \dot{}, c_{06}{}^6 = (\ln \omega_2^1) \dot{} \\ c_{ij}{}^A &= 0 \quad (i, j = 1, 2, 3, A = 0, \dots, 6), \quad c_{45}{}^6 = -\frac{\omega_3^2 \omega_3^1}{\omega_2^1}, \quad c_{56}{}^4 = -\frac{\omega_3^1 \omega_2^1}{\omega_3^2}, \quad c_{46}{}^5 = -\frac{\omega_3^2 \omega_2^1}{\omega_3^1} \\ c_{14}{}^A &= 0 \quad (A = 0, \dots, 6), \quad c_{15}{}^3 = -\frac{f_1}{f_3} \omega_3^1, \quad c_{16}{}^2 = -\frac{f_1}{f_2} \omega_2^1 \\ c_{24}{}^3 &= -\frac{f_2}{f_3} \omega_3^1, \quad c_{25}{}^A = 0 \quad (A = 0, \dots, 6), \quad c_{26}{}^1 = \frac{f_2}{f_1} \omega_2^1, \\ c_{34}{}^2 &= \frac{f_3}{f_2} \omega_3^1, \quad c_{35}{}^1 = \frac{f_3}{f_1} \omega_3^1, \quad c_{36}{}^A = 0 \quad (A = 0, \dots, 6). \end{aligned}$$